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# ON CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS BY CONDITIONAL EXPECTATIONS OF UPPER RECORD VALUES

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Abstract. In this paper, general classes of continuous distributions are characterized by considering the conditional expectations of functions of upper record statistics. The specific distribution considered as a particular case of the general class of distribution are Exponential, Exponential Power(EP), Inverse Weibull, Beta Gumbel, Modified Weibull(MW), Weibull, Pareto, Power, Singh-Maddala, Gumbel, Rayleigh, Gompertz, Extream value 1, Beta of the first kind, Beta of the second kind and Lomax.

## 1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say $X_j$ is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. By definition, $X_1$ is an upper record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

Lee(2003) showed that $X \in PAR(\theta)$ if and only if $(\theta+1)^i E[X_{U(n+i)} \mid X_{U(m)} = y] = \theta^i E[X_{U(n)} \mid X_{U(m)} = y]$ for $i = 1, 2, 3,\ n \geq m+1$.

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Also, Faizan, Khan and Haque(2010) showed that

$$E[h(X_{U(s)}) - h(X_{U(r)}) \mid X_{U(m)} = x] = (s-r)c$$

if and only if $\bar{F}(x) = e^{-\frac{(h(x))}{c}}$, $c > 0$, where $h(x)$ is a monotonic and differentiable function of $x$ and $m \leq r < s$.

In this paper we will give characterizations of the continuous distributions by using equivalence between the adjacent conditional expectations of upper record values.

## 2. Main results

THEOREM 2.1. *Let $X$ be an absolutely continuous random variable with the cdf $F(x)$ and the pdf $f(x)$ on the support $(\alpha, \beta)$, where $\alpha$ and $\beta$ may be finite or infinite. Then, for $a > 0$, $k > 0$,*

$$\begin{aligned}(a+1)E[(g(X_{U(n+1)}))^k \mid X_{U(m)} = y] \\ = aE[(g(X_{U(n)}))^k \mid X_{U(m)} = y]\end{aligned} \tag{2.1}$$

*if and only if*

$$F(x) = 1 - (g(x))^{ak}, \tag{2.2}$$

*where $g(x)$ is a monotonic and differentiable function of $x$ such that $g(x) \to 1$ as $x \to \alpha$ and $g(x) \to 0$ as $x \to \beta$.*

*Proof.* For the necessity part, it is easy to see that (2.2) implies (2.1).

For the sufficiency part, using Ahsanullah formula(1995), we get the following equation

(2.3)

$$\begin{aligned}&\frac{a+1}{1-F(y)} \int_y^\infty \frac{1}{(n-m)!}\left(ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} (g(x))^k f(x)dx \\ &= \frac{a}{1-F(y)} \int_y^\infty \frac{1}{(n-m-1)!}\left(ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} (g(x))^k f(x)dx.\end{aligned}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both sides of (2.3) with respect to $y$ and simplify to obtain the following equation

$$\frac{-f(y)}{1-F(y)} = \frac{ak(g(y))^{k-1}g'(y)}{(g(y))^k} \tag{2.4}$$

Integrating both sides of (2.4) with respect to $y$, we get $F(y) = 1 - (g(y))^{ak}$. Hence, the theorem is proved. $\square$

Remark 2.2. A number of distributions can be characterized by a proper choice of $ak$ and $g(x)$.

Table 1. Examples based on the cdf $F(x) = 1 - (g(x))^{ak}$

| Distribution | $F(x)$ | $ak$ | $g(x)$ |
|---|---|---|---|
| Exponential | $1 - e^{-\lambda x},\ 0 < x < \infty$ | $ak = \lambda$ | $e^{-x}$ |
| Weibull | $1 - e^{-\lambda x^p},\ 0 < x < \infty$ | $ak = \lambda$ | $e^{-x^p}$ |
| Pareto | $1 - x^{-\lambda},\ 1 < x < \infty$ | $ak = \lambda$ | $x^{-1}$ |
| Beta 1st kind | $1 - (1-x)^{\lambda},\ 0 < x < 1$ | $ak = \lambda$ | $(1-x)$ |
| Beta 2nd kind | $1 - (1+x)^{-\lambda},\ 0 < x < \infty$ | $ak = \lambda$ | $1/(1+x)$ |
| Beta Gumbel | $1 - (1 - exp[-e^{-x}])^{\lambda}$ $-\infty < x < \infty$ | $ak = \lambda$ | $1 - exp[-e^{-x}]$ |
| Lomax | $1 - (1 + \frac{x}{\alpha})^{-1},\ 0 < x < \infty$ | $ak = 1$ | $1/(1 + \frac{x}{\alpha})$ |
| Singh-Maddala | $1 - (1 + \theta x^p)^{-\lambda},\ 0 < x < \infty$ | $ak = \lambda$ | $1/(1 + \theta x^p)$ |
| Kappa | $\frac{x^p}{\lambda + x^p},\ 0 < x < \infty$ | $ak = 1$ | $\frac{\lambda}{\lambda + x^p}$ |
| Gompertz | $1 - exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ $0 < x < \infty$ | $ak = 1$ | $exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ |
| Rayleigh | $1 - exp[-2^{-1}\theta^{-2}x^2]$ $0 < x < \infty$ | $ak = 1$ | $exp[-2^{-1}\theta^{-2}x^2]$ |
| MW | $1 - exp[-\lambda x^{\alpha} e^{\beta x}],\ 0 < x < \infty$ | $ak = \lambda$ | $exp[-\lambda x^{\alpha} e^{\beta x}]$ |
| EP | $1 - exp[1 - e^{x^{\beta}}],\ 0 < x < \infty$ | $ak = 1$ | $exp[1 - e^{x^{\beta}}]$ |
| Extream value I | $1 - exp[-e^{x}],\ -\infty < x < \infty$ | $ak = 1$ | $exp[-e^{x}]$ |
| Gumbel | $exp[-e^{-x}],\ -\infty < x < \infty$ | $ak = 1$ | $(1 - exp[-e^{-x}])$ |

Theorem 2.3. *Let $X$ be an absolutely continuous random variable with the cdf $F(x)$ and the pdf $f(x)$ on the support $(\alpha, \beta)$, where $\alpha$ and $\beta$ may be finite or infinite. Then, for $k > r > 0$,*

$$E[(g(X_{U(n+1)}))^k \mid X_{U(m)} = y] = E[(g(X_{U(n)}))^{k-r} \mid X_{U(m)} = y] \tag{2.5}$$

*if and only if*

$$F(x) = 1 - (1 + \frac{1}{(g(x))^r})^{-(\frac{k}{r} - 1)} \tag{2.6}$$

*where $g(x)$ is a monotonic and differentiable function of $x$ such that $1/(g(x))^r \to 0$ as $x \to \alpha$ and $1/(g(x))^r \to \infty$ as $x \to \beta$.*

*Proof.* For the necessity part, it is easy to see that (2.6) implies (2.5).
For the sufficiency part, we get

(2.7)

$$\frac{1}{1-F(y)}\int_y^\beta \frac{1}{(n-m)!}\left(ln\frac{1-F(y)}{1-F(x)}\right)^{n-m}(g(x))^k f(x)dx$$
$$=\frac{1}{1-F(y)}\int_y^\beta \frac{1}{(n-m-1)!}\left(ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1}(g(x))^{k-r} f(x)dx.$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both sides of (2.7) with respect to $y$ and simplify to obtain the following equation

$$\frac{-f(y)}{1-F(y)}=\frac{(k-r)(g(y))^{-r-1}g'(y)}{1+(g(y))^{-r}}. \tag{2.8}$$

Integrating both sides of (2.8) with respect to $y$, we get $F(y) = 1-(1+\frac{1}{(g(y))^r})^{-(\frac{k}{r}-1)}$, for $k>r>0$.

Hence, the theorem is proved. □

TABLE 2. Examples based on the cdf $F(x)=1-(1+\frac{1}{(g(x))^r})^{-(\frac{k}{r}-1)}$

| Distribution | $F(x)$ | $k, r$ | $1/(g(x))^r$ |
|---|---|---|---|
| Exponential | $1-e^{-\lambda x},\ 0<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $e^x-1$ |
| Weibull | $1-e^{-\lambda x^p},\ 0<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $e^{x^p}-1$ |
| Pareto | $1-x^{-\lambda},\ 1<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $x-1$ |
| Beta 2nd kind | $1-(1+x)^{-\lambda}$ $0<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $x$ |
| Lomax | $1-(1+\frac{x}{a})^{-\lambda}$ $0<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $\frac{x}{a}$ |
| Singh-Maddala | $1-(1+\theta x^p)^{-\lambda}$ $0<x<\infty$ | $\frac{k}{r}-1=\lambda$ | $\theta x^p$ |
| EP | $1-exp[1-e^{x^\beta}]$ $0<x<\infty$ | $\frac{k}{r}=2$ | $exp[-1+e^{x^\beta}]-1$ |
| Gompertz | $1-exp[-\frac{\lambda}{\mu}(e^{\mu x}-1)]$ $0<x<\infty$ | $\frac{k}{r}=2$ | $exp[\frac{\lambda}{\mu}(e^{\mu x}-1)]-1$ |
| Rayleigh | $1-exp[-2^{-1}\theta^{-2}x^2]$ $0<x<\infty$ | $\frac{k}{r}=2$ | $exp[2^{-1}\theta^{-2}x^2]-1$ |
| MW | $1-exp[-\lambda x^\alpha e^{\beta x}]$ $0<x<\infty$ | $\frac{k}{r}=2$ | $exp[\lambda x^\alpha e^{\beta x}]-1$ |
| Extream value I | $1-exp[-e^x]$ $-\infty<x<\infty$ | $\frac{k}{r}=2$ | $exp[e^x]-1$ |

Remark 2.4. A number of distributions can be characterized by a proper choice of $k, r$ and $g(x)$.

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